ISRAEL JOURNAL OF MATHEMATICS **165** (2008), 211–231
DOI: 10.1007/s11856-008-1010-5

TIGHT UPPER BOUNDS ON THE NUMBER OF INVARIANT COMPONENTS ON TRANSLATION SURFACES

BY

Yoav Naveh

Department of Mathematics, Ben Gurion University
Be'er Sheva, Israel 84105
e-mail: navehy@math.bgu.ac.il

ABSTRACT

An abelian differential on a surface defines a flat metric and a vector field on the complement of a finite set of points. The vertical flow that can be defined on the surface has two kinds of invariant closed sets (i.e. invariant components) — periodic components and minimal components. We give upper bounds on the number of minimal components, on the number of periodic components and on the total number of invariant components in every stratum of abelian differentials. We also show that these bounds are tight in every stratum.

1. Introduction and statement of the results

Let S be a compact oriented surface with complex structure, and Φ a holomorphic abelian differential on S having zeros at $\Sigma = \{p_1, \ldots, p_n\}$. A comprehensive description of such surfaces, also called translation surfaces, can be found in [EMZ]. The pair (S, Φ) admits the following geometric structure — on $S\backslash\Sigma$ there is an atlas of open sets and coordinate charts such that the transition functions defined on the overlaps are translations. This results in a flat metric on $S\backslash\Sigma$ with cone-type singularities at points in Σ, and whose associated holonomy group on $S\backslash\Sigma$ is trivial. The angles around these cone-type singularities are integer multiple of 2π, and one can define a vector field on $S\backslash\Sigma$. Following

Received April 20, 2006 and in revised form November 13, 2006

[EMZ] we call a surface with such a flat structure a flat surface. Denoting the pre-image of the vertical direction in the plane the vertical direction, there is a well-defined vertical foliation on $S \backslash \Sigma$. A critical leaf of the foliation is a leaf coming out of a singular point. On the complement of the critical leaves a flow in the vertical direction can be defined (called the vertical flow). When we refer to a flat surface we always assume that a vertical direction (and hence a vertical flow) is specified.

It is known that the vertical flow decomposes a flat surface into a finite union of invariant components, which can be either periodic or minimal, and that the boundaries of those components are comprised of saddle connections. Call a point p on S generic, if p is not on a critical leaf (i.e. the orbit of p under the vertical flow is defined). If p is a generic point whose orbit under the vertical flow is a closed geodesic (i.e. periodic point), nearby points are also periodic with orbits that are parallel to the original closed geodesic and of the same length. A maximal closed connected subset of closed geodesics is a periodic component. Its boundary is a closed curve comprised of saddle connections (possibly one closed saddle connection), with the internal angles inside the periodic component, between the outgoing and incoming edges, exactly π (the converse is also true; such a boundary curve is the boundary of a periodic component). We call a closed curve comprised of saddle connections in the vertical direction a vertical circle. A minimal component is the closure of the orbit of a generic, nonperiodic point. The intersection between a minimal component and a horizontal segment on the flat surface is a finite union of nondegenerate closed intervals (this is due to [Bo]), so a minimal component has “nonzero width” at any point. The restriction of the flow to a minimal component is a minimal flow — every semi-orbit (of a generic point) is dense.

Let α be a partition of $2g-2$ (i.e. a representation of $2g-2$ as an unordered sum of positive integers). Let $\mathcal{H}(\alpha)$ denote the moduli space of pairs (S, Φ) where S is a compact surface of genus g with complex structure and Φ a holomorphic abelian differential on S such that the orders of its zeros are given by α. A zero of order a corresponds to an angle of $(2a+2)\pi$ around the singular point. $\mathcal{H}(\alpha)$ is also called a stratum.

The problem this paper deals with is finding optimal upper bounds in each stratum on the number of minimal components, on the number of periodic components, and on the total number of invariant components. We state two theorems solving this problem completely. The first theorem deals with minimal components, and the second deals with all invariant components, and in particular with periodic components.

THEOREM 1: *Let $\mathcal{H} = \mathcal{H}(a_1, a_2, \ldots, a_j)$ be a stratum of surfaces of genus g.*

(1) *If $a_i \leq g-1$, $i = 1, 2, \ldots, j$, then for every flat surface in $\mathcal{H}$ an upper bound on the number of minimal components is g, and this bound is tight.*
(2) *Otherwise, for every flat surface in $\mathcal{H}$ an upper bound on the number of minimal components is $g-1$, and this bound is tight.*

Remark 1.1: The stratum $\mathcal{H}(\emptyset)$ is considered as a stratum fulfilling the first condition.

Let $\mathbf{M}$ denote the number of minimal components, and $\mathbf{P}$ denote the number of periodic components on a flat surface.

THEOREM 2: *Let $\mathcal{H} = \mathcal{H}(a_1, a_2, \ldots, a_j)$ be a stratum of surfaces of genus $g \geq 2$. Denote $B = \{i : a_i \text{ is odd}\}$. Fix $0 \leq \mathbf{M} \leq g-1$ and denote $m = \max\{0, \mathbf{M} - [g-1-|B|/2]\}$, then for every flat surface in $\mathcal{H}$*

$$\mathbf{M} + \mathbf{P} \leq g - 1 + j - m,$$

and this bound is tight.

If $\mathbf{M} = g$, then $\mathbf{P} = 0$ and $\mathbf{M} + \mathbf{P} = g$.

Where $|\cdot|$ means the number of elements in a set. The upper bound in the case $\mathbf{M} = 0$, was proved by J. Smillie.

Remark 1.2: In the last theorem, the meaning of the bound being tight in a stratum, is that for every $\mathbf{M}$ (between 0 and $g-1$) there is a surface in that stratum that causes the inequality to be an equality.

We prove the upper bounds in Section 2 and show the bounds are tight in Section 3.

ACKNOWLEDGMENTS. I would like to thank my instructor Barak Weiss for his constant help, support and encouragement during the whole time of our mutual work.

I thank Yair Minsky, John Smillie and Patrick Hooper for useful discussions, and also the anonymous referee whose comments greatly improved the presentation. I thank the Binational Science Foundation and the Israel Science Foundation for the financial support. This paper is the main part of the author's M.Sc. thesis at Ben Gurion University, under the supervision of Barak Weiss.

2. Upper bounds

Before we go on and prove the upper bounds, we want to establish a general equality that connects the total angle around all singular points on a flat surface with its Euler characteristic. This equality is a slight generalization of a known equality resulting from Gauss–Bonnet formula

$$\sum_{i=1}^{j}(k_i - 2) = 4g - 4, \tag{2.1}$$

where S is a flat surface (without boundary) of genus g with angles $k_1, k_2, \ldots, k_j$ at the singular points.

In this paper we consider also surfaces with boundary. We call an oriented surface with boundary S a flat surface if S has a flat structure, and, in addition, its boundary curves are vertical (vertical circles or closed vertical geodesics). Note that we allow singular points on the boundary with internal angles multiple of π.

From here on we will use the following terminology and notations

(1) We define two types of singular points on a flat surface with boundary. Singular points of type 1 are singular points which are not on the boundary of the flat surface. The total number of such points is denoted by $j^{(1)}$. Singular points of type 2 are singular points which are on the boundary of the flat surface. The total number of such points is denoted by $j^{(2)}$. We have $j = j^{(1)} + j^{(2)}$, where j is the total number of singular points.

(2) $\mathbf{C}$ will denote the number of connected components of a surface (when each connected subsurface is a flat surface with boundary). The genus of a non-connected surface will be the sum of genera of the connected components.

(3) **B** will denote the number of boundary components. By our early definition each boundary component is comprised of vertical circles or closed vertical geodesics.

(4) When we refer to an angle k_i , we mean an angle of $k_i\pi$. If (S,Φ) is a flat surface in $\mathcal{H}(a_1,a_2,\ldots,a_j)$, then by k_i we refer to the angle around the zero of order a_i, and $k_i = 2a_i + 2$.

(5) Let S be a flat surface with boundary, with angles of $k_1, k_2, \ldots, k_j$. Note that some of these points can be singular points of type 2, in which case the angle is measured when circling around the singular point and inside the surface. We denote by I the quantity

$$I = \sum_{i=1}^{j^{(1)}} (k_i - 2) + \sum_{i=1}^{j^{(2)}} (k_i - 1).$$

Remark 2.1: The meaning of internal angle explained in (5) is also applicable in the following case: if L is an invariant component on a flat surface, and p a singular point on the boundary of L, then the internal angle of p in L is the angle that is measured when circling around p and inside L.

The following lemma is well-known.

LEMMA 2.2: *Let S_1 be a (not necessarily connected) flat surface with boundary of genus g. Then*

$$I = 4g - 4 + 2\mathbf{B} - 4(\mathbf{C} - 1).$$

If S_1 is connected then the equality becomes

$$I = 4g - 4 + 2\mathbf{B}. \tag{2.2}$$

Remark 2.3: Equation (2.2) in a more familiar form (as a version of the Gauss–Bonnet formula) is obtained by substituting $4g - 4 + 2\mathbf{B}$ with -2χ, where χ is the Euler characteristic of the surface, and multiplying both sides by $-\pi$.

Let us define how to cut flat surfaces along vertical curves. Let β be a vertical curve on a flat surface S. The resulting flat surface after cutting S along β is the completion of $S\backslash\beta$ by the path metric. It is important to note that although the boundary of an invariant component can consist of two vertical circles with a common singular point (like Figure 8), and more complicated configurations of this sort, after cutting a flat surface along any configuration of vertical curves, its boundary will be comprised out of disjoint vertical circles and/or closed geodesics (i.e. every connected boundary component is either a vertical circle or a closed geodesic).

Since we were unable to find a suitable reference, we include the following sketch of a proof.

Sketch of proof of Lemma 2.2. Since all the quantities are additive, we can restrict ourselves to the connected case. If S_1 has no boundary, then the claim follows from equality (2.1). If S_1 is a flat surface with boundary, we can "close" it to a flat surface without boundary S, by identifying its boundary with the boundary of its mirror image. For S the claim holds, and now we separate S back to S_1 and its mirror image and record the change in all the quantities present in the lemma. In particular, I remains unchanged. Since all the quantities of S_1 are exactly half of the corresponding quantities on S, the equality for S_1 is easily established. ∎

The following proposition is the main step in proving Theorem 1.

PROPOSITION 2.4: *Let S be a flat surface with boundary of genus g, then an upper bound on the number of minimal components is g.*

Proof. The proof will be by induction on the genus g. Assume $g = 0$. If S has no boundary there is nothing to prove, as there is no abelian differential over the sphere. Assume, by contradiction, that S is a flat surface of genus zero and there is at least one minimal component on S. If there is more than one invariant component, we cut along the boundary of the minimal component and are left with a genus zero flat surface with exactly one minimal component. By Lemma 2.2 we have

$$I = 4g - 4 + 2\mathbf{B} = -4 + 2\mathbf{B},$$

for S. Every vertical circle on the boundary contains at least one singular point. If there is exactly one singular point, then the boundary is a closed saddle connection, and its incoming and outgoing directions at the singular point are opposite. Thus the internal angle at the singular point is of the form $(2n+1)\pi$, $1 \leq n \in \mathbb{N}$ (if n equals 0 the invariant component must be periodic), and so is at least 3π. If there is more than one singular point, each has an internal angle of at least 2π. From this we conclude that each vertical circle on the boundary of a minimal component adds at least 2 to I, and so

$$2\mathbf{B} \leq I = -4 + 2\mathbf{B},$$

a contradiction.

In the general case, let g be greater then zero, and S be a flat surface with boundary. If S has one minimal component there is nothing to prove. Otherwise, as there is more than one invariant component, there must be a boundary curve of a minimal component to cut along. Note that although the boundary of an invariant component is comprised of vertical circles, some edges (i.e. saddle connections) — but not all — on the boundary curve along which we cut can be on the boundary of S. After the cut, we fall into exactly one of the two following cases:

(1) The resulting flat surface $\widetilde{S}$ is connected.

Then $\widetilde{g} = g - 1$ and by the induction hypothesis, the number of minimal components on $\widetilde{S}$ is at most $\widetilde{g}$. But this is exactly the number of minimal components on the original flat surface.

(2) There are two connected flat surfaces S_1 and S_2 of genera g_1, g_2, and $g_1 + g_2 = g$.

If $g_1, g_2 > 0$, then $g_1, g_2 < g$, and using the induction hypothesis and the additivity of the number of minimal components, the claim follows. If, for example, $g_1 = 0$, then by the base case, S_1 has no minimal components. So we just cut again until we fall into one of the first two cases (this process is, of course, finite). ∎

As one can see from the last proof, if the curve one cuts along does not separate the flat structure into two connected components, the bound that results is actually better: $g - 1$. We now analyze this phenomenon in more detail.

Definition 2.5: Let (S, Φ) be a flat surface. Let $\widehat{S}$ be the universal cover of $S \backslash \Sigma$, where Σ is the collection of singular points. There is an isometric map of $\widehat{S}$ to $\mathbb{R}^2$ called the developing map, which maps a lift $\tilde{\alpha}$ of an oriented curve α on (S, Φ) to a curve in $\mathbb{R}^2$. We denote by $hol(\alpha)$ the difference of the endpoints of the image. The vector $hol(\alpha) \in \mathbb{R}^2$ is called the holonomy vector of α.

The following argument appears in [EMZ]. In term of the atlas of charts defining the flat structure, the corresponding abelian differential has the form $\omega = dz = dx + idy$. This has an important consequence: for any oriented curve β on (S, Φ), the holonomy vector of β coincide with the integral $\int_\beta \omega$ of ω over β. Since the 1-form is closed, it means, in particular, that if β and α are homologous, then $hol(\beta) = hol(\alpha)$ (when a path β joins a pair of distinct points we say that a path α is homologous to β if it joins the same pair of points and if the closed loop $\beta \cdot \alpha^{-1}$ is homologous to zero, that is, breaks the surface into two components).

Combining the observations made in the last paragraph with Proposition 2.4 we obtain

COROLLARY 2.6: *If a flat surface (S, Φ) of genus g has a vertical circle or closed vertical geodesic that is not homologous to zero then an upper bound on the number of minimal components is $g - 1$.*

Proof. Cutting along such a curve keeps the flat surface connected. ∎

COROLLARY 2.7: *If a flat surface (S, Φ) of genus g has a periodic component then an upper bound on the number of minimal components is $g - 1$. This proves the last statment in Theorem 2.*

Proof. The integral over a closed geodesic in the interior of the periodic component cannot be zero, so cutting along such a curve keeps the flat surface connected. ∎

In order to complete the proof of the upper bounds stated in Theorem 1 we have to show that if (S, Φ) is a flat surface over a surface of genus g in $\mathcal{H}(a_1, a_2, \ldots, a_j)$ with g minimal components, then $a_i \leq g - 1,\ i = 1, 2, \ldots, j$.

Definition: A **slit** on a flat surface is obtained by cutting along a vertical segment $\overline{pp'}$ in the interior of an invariant component. The surface now has an additional boundary component which is a vertical circle with two antipodal singular points p and p'. Denote the two saddle connection comprising the slit γ_1 and γ_2. If we choose two points, one on γ_1 and the other on γ_2 , equidistant from p and identify them we get a *slit with two points identified* (see Figure 1).

Remark 2.8: A boundary of a flat surface cannot be a slit with identified points, as this does not even qualify as a surface with boundary. However, a slit with identified points can occur as a boundary of an invariant component inside a flat surface. This will be the case we will refer to in the proof.

Proof of Theorem 1. Let (S, Φ) be a flat surface of genus g in

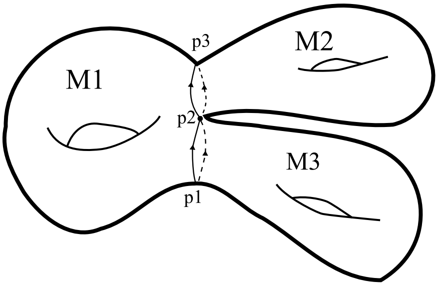


Figure 1. A genus 3 flat surface with three invariant components in $\mathcal{H}(2,1,1)$. The boundary of $M1$ is a slit with identified points. The boundaries of $M2$ and $M3$ are slits.

$\mathcal{H}(a_1, a_2, \ldots, a_j)$ and assume that (S, Φ) has g minimal components. By Corollary 2.6 all the boundary curves are homologous to zero. Cutting along all the boundary curves we get g sub-surfaces with boundary $N_1, N_2, \ldots, N_g$ and each N_i has exactly one invariant component which is minimal. By Proposition 2.4 the genus g_i of N_i is equal to or greater than one. Since $\sum_{i=1}^{j} g_i = g$, we have $g_i = 1$ for all i. By Lemma 2.2

$$I_i = 4g_i - 4 + 2\mathbf{B}_i = 2\mathbf{B}_i\,, \quad i = 1, 2, \ldots, g. \tag{2.3}$$

By a previous argument (proof of Proposition 2.4, base case) each boundary component of a minimal components adds at least two to I, and from (2.3) it follows that each boundary component adds exactly two to I. Since every boundary component has at least one singular point there are only two options for each boundary curve:

(1) There is one singular point with an internal angle of 3π.
(2) There are two singular points each with an internal angle of 2π.

Option (1) is not possible as such a boundary curve is not homologous to zero. In option (2) note that the two singular points need to be antipodal for the boundary curve to be homologous to zero. So all boundary components on our sub-surfaces look the same, they are slits. It is important to note that on (S, Φ) a minimal component can have a slit with identified points as boundary, but after the cuts are performed a slit with identified points turns into a slit.

To conclude: if (S,Φ) has g minimal components, then all of them are tori with either slits or slits with identified points and all slits are homologous to zero. In particular, all singular points are on the boundaries of minimal components.

If L is a minimal component and p a singular point on the boundary of L then the internal angle around p in L is at most 2π (see Lemma 2.9 below). If so, the maximal angle around any singular point on (S,Φ) is $2g\pi$, and the maximal order of a zero is $g-1$. This completes the proof (of the upper bounds) of Theorem 1. ∎

Lemma 2.9: *Let (S,Φ) be a flat surface of genus g with g minimal components. If L is a minimal component and p a singular point on the boundary of L then the internal angle around p in L is at most 2π.*

Proof. The lemma follows directly from the fact that all the minimal components are tori with slits and/or slits with identified points (as explained in the previous proof). ∎

In order to prove Theorem 2, we will first prove a weaker claim.

Proposition 2.10: *Let (S,Φ) be a flat surface of genus $g \geq 2$ in a stratum $\mathcal{H}(a_1, a_2, \ldots, a_j)$, then an upper bound on the total number of invariant components on (S,Φ) is $g-1+j$. That is, if $\mathbf{M}$ is the number of minimal components, $\mathbf{P}$ the number of periodic components on (S,Φ), then $\mathbf{M}+\mathbf{P} \leq g-1+j$.*

Proof. The proof is based on an argument of J. Smillie.

If $\mathbf{P}=0$, by Theorem 1 the inequality holds. Let $\gamma_1, \gamma_2, \ldots, \gamma_r$ be core curves (i.e. closed vertical geodesics in the interior of periodic components) in the $\mathbf{P}$ periodic components. Cutting along these curves results in $N_1, N_2, \ldots, N_s$ sub-surfaces. Cutting along each curve creates two boundary components, so the total number of boundary components on these sub-surfaces is $2\mathbf{P}$. Let $\mathbf{B}_i$ denote the number of boundary components on a sub-surface N_i. The Euler characteristic of N_i is $\chi(N_i) = 2-2g_i-\mathbf{B}_i$, so $\mathbf{B}_i = 2-2g_i-\chi(N_i)$. Summing over the sub-surfaces N_i and using the additivity of the Euler characteristic we get

$$\mathbf{P} = \frac{1}{2}\sum_{i=1}^{s}\mathbf{B}_i = \frac{1}{2}\sum_{i=1}^{s}(2-2g_i-\chi(N_i)) = s - \frac{1}{2}\chi(S) - \sum_{i=1}^{s} g_i. \tag{2.4}$$

The sub-surfaces $N_1, N_2, \dots, N_s$ contain the $\mathbf{M}$ minimal components. By Proposition 2.4, if there are $\mathbf{M}_i$ minimal components on a sub-surface N_i then $\mathbf{M}_i \leq g_i$, so $\mathbf{M} \leq \sum_{i=1}^{s} g_i$. Plugging into (2.4) we get

$$\mathbf{P} \leq s - \frac{1}{2}\chi(S) - \mathbf{M} = s + g - 1 - \mathbf{M}$$

or

$$\mathbf{M} + \mathbf{P} \leq s + g - 1. \tag{2.5}$$

We will now bound s. Each boundary curve of every invariant component contains a singular point, so every sub-surface contains at least one singular point. Therefore, $s \leq j$ and the proof is complete. ∎

Proof of Theorem 2. In case $m = 0$ the theorem follows from Proposition 2.10. We now consider the case where $m > 0$.

Let us define the set B (from Theorem 2) and its complement set B_{comp} in term of the angles k_i rather than the orders of the zeros a_i (remember that $k_i = 2a_i + 2$). Denote $B = \{i : k_i = 4n_i, n_i \in \mathbb{N}\}$ and $B_{comp} = \{i : k_i = 4n_i + 2, n_i \in \mathbb{N}\}$, where the stratum is $\mathcal{H}(a_1, a_2, \dots, a_j)$.

It follows from equality (2.1) that

$$g - 1 - \frac{|B|}{2} = \sum_{i \in B_{comp}} n_i + \sum_{i \in B} (n_i - 1) = \mathbf{M} - m\ . \tag{2.6}$$

In the previous proof we saw that $\mathbf{M} + \mathbf{P} \leq s + g - 1$. We will now show that $s \leq j - m$.

Let L be a minimal component on a flat surface with more than one invariant component, and look at the boundary of L. It consists of a union of vertical circles, and each circle contains a singular point. If L has exactly one singular point, which must be on its boundary, L has exactly one connected boundary component. This connected boundary component of L is a union of vertical circles whose intersection is exactly the singular point (denoted by p). The circles are cyclically arranged around the sole singular point, and each vertical circle is a closed saddle connection (topologically, this looks like a flower, where the petals are the closed saddle connections). As this is the only boundary component of L, this boundary component consists of at least two closed saddle connections, and so at least two internal angles around p (one closed saddle connection is not homologous to zero, which is impossible by the beginning of this sentence). As this boundary curve is homologous to zero, there are at least two even internal angles around p (an even angle is an angle of an even multiple of π). To conclude, a minimal component with exactly one singular point on its boundary (and no singular points in its interior) has a total angle of at least 4π around the singular point.

Let p be a singular point on S with an angle of $k\pi$. We will bound the number x of minimal components L such that p lies on the boundary of L, and the boundary of L contains no other singular points. It follows from the previous paragraph that if $k = 4n + 2$, then x is at most n, and we claim that if $k = 4n$ then x is at most $n - 1$. This can be explained in two ways.

First, if there are exactly n such minimal components around p, these are all the invariant components around p. But then, the neighborhood of p is not homeomorphic to a disc. This can be visualized if you start with n minimal tori with slits glued to one another in cyclic order - the tori are glued around two singular points (the upper and lower ends of the slits), and then identify the two singular points on the flat surface. Second, if this is the case then p must be the only singular point on the flat surface: p is the only singular point contained in those n minimal components. If we assume that there are more singular points on the flat surface then these singular points are in different components. But these different components cannot be connected to our initial n minimal components as each boundary between invariant components must contain a singular point, so this is impossible. Now, if p is the only singular point on the flat surface then $k-2 = 4n-2 \neq 0 \pmod 4$, and by equality (2.1) this is impossible. We already know that $s \leq j$. Let us compute the maximal number of minimal components with exactly one singular point (on the boundary) that can occur on a flat surface (S, Φ) in a stratum $\mathcal{H}(a_1, a_2, \ldots, a_j)$. By all that was said, there can be no more than

$$\sum_{i \in B_{comp}} n_i + \sum_{i \in B} (n_i - 1)$$

such minimal components on (S, Φ). If so, there are at least m minimal components (m as can be extracted from equality (2.6)) with at least two singular points in them. The collection of singular points on the sub-surfaces $N_1, N_2, \ldots, N_s$ gives a partition of the set of all singular points into s nonempty subsets. The last statement indicated that at least m of these subsets contain two or more singular points. So $s \leq j - m$ and plugging it into (2.5) gives

$\mathbf{M} + \mathbf{P} \leq g - 1 + j - m$. This completes the proof of the upper bounds of Theorem 2. ∎

3. Constructions realizing the bounds

The general scheme in this section will be to present specific constructions in low genera and then use induction to generalize these constructions.

For our concrete constructions we will use a small number of "building blocks," that are defined formally below. Some of the constructions are "zippered rectangles" constructions, defined by Veech (see [Ve]). In short, this construction involves a suspension over interval exchange transformations (iet for short)—let T be an iet with a permutation Π and a length vector $X = (X_1, X_2, \ldots, X_n)$. For each sub-interval X_i of an iet T, a rectangle is defined. The bottom edge of the rectangle is identified with X_i, the upper edge is identified with the image of X_i by T and the sides are identified with each other in a way dictated by the permutation of T. This results in a compact surface with an abelian differential. Following Veech, the permutation of the iet (and the number of intervals) define a specific stratum to which the corresponding zippered rectangle belongs. Beside the permutation there are 3 n-tuples (for an n interval iet) that determine a construction:

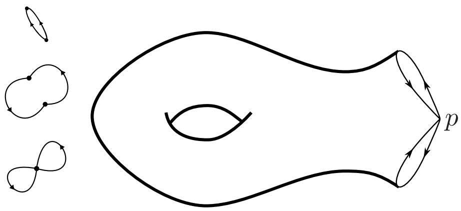
(1) $X = (X_1, X_2, \ldots, X_n)$. These are the lengths of the sub-intervals of the iet.
(2) $h = (h_1, h_2, \ldots, h_n)$. These are the heights of the rectangles.
(3) $a = (a_1, a_2, \ldots, a_n)$. These are the the locations of the singular points on the surface. Roughly, it defines up until which points adjacent rectangles will be identified one to the other ("zippered together").

All this data is subjected to a set of equalities and a set of inequalities as described in [Ve].

BUILDING BLOCKS.

Slit torus: Take a flat torus T and cut it along a vertical slit of length d. A slit torus can be minimal or periodic.

Figure-8 torus: Take a slit torus and identify the two type 2 singular points on the boundary. The resulting surface has one boundary component with one singular point (denote it p) and two closed (vertical) saddle connections of equal length. The total internal angle around p is 4π. A figure-8 torus can be minimal or periodic (see figure 2).



(a) (b)

Figure 2. In (a) (left) we illustrate the process of turning a slit to a figure-8 boundary. In (b) (right) there is a figure-8 torus.

We now prove the first case in Theorem 1.

In the case of $\mathcal{H}(\emptyset)$, a minimal torus realizes the bound of $g = 1$.

PROPOSITION 3.1: *Let $\mathcal{H} = \mathcal{H}(a_1, a_2, \ldots, a_j)$ be a stratum of surfaces of genus $g \geq 2$, such that $a_i \leq g - 1,\ i = 1, 2, \ldots, j$. Then there exists a flat surface in $\mathcal{H}$ with g minimal components.*

Moreover, for every $1 \leq k < l \leq j$ there exists a flat surface (S, Φ) with g minimal components and with two singular points p_k, p_l of orders a_k, a_l respectively such that there is a saddle connection between p_k and p_l on (S, Φ).

Before we continue to the proof we need to establish the following claim.

LEMMA 3.2: *Let (S, Φ) be a flat surface of genus $g \geq 2$ with g minimal components, p a singular point on (S, Φ). Then there exists a minimal component L, with p on its boundary such that the boundary component of L containing p is a slit (and p the top or bottom end of that slit).*

Proof. The proof is by induction on the genus g. From the proof of Theorem 1 in the previous section, we know that all the minimal components on (S, Φ) are tori with slits or slits with identified points and all the slits are homologous to zero. If $g = 2$, the only stratum is $\mathcal{H}(1, 1)$ and (S, Φ) must be two slit tori glued along the slits, so the claim holds. In the general case, there must be a minimal component on (S, Φ) which is a torus with a single slit (this is easily proved by another induction). If p is on the boundary of this torus we are done. Otherwise, cut this torus from (S,Φ) and close the slit on the remaining surface. The resulting flat surface is of genus $g-1$ and we apply the induction hypothesis. Since the neighborhood of p is not effected by this cutting process the minimal component L also exists on (S,Φ). ∎

Proof of Proposition 3.1. The proof is by induction on the genus g.

If $g=2$ the only stratum to consider is $\mathcal{H}(1,1)$. If we take two minimal slit tori and glue them one to the other along the slits, we get the desired flat surface in $\mathcal{H}$ (there is only one choice for k and l).

Now assume that $g \geq 3$. Let $\mathcal{H}(a_1, a_2, \ldots, a_j)$ be a stratum as described in the proposition (over a surface of genus $g \geq 3$). Fix k, l, $1 \leq k < l \leq j$. We separate into two cases

- $\mathcal{H}$ is of the form $\mathcal{H}(g-1, a_2, a_3, \ldots, a_j)$.

  We describe explicitly how to build a flat surface meeting all the conditions stated in the proposition.

  First, we take a_2+1 minimal slit tori and glue them together in cyclic order: each slit consists of two saddle connections a left one and a right one. Glue the left saddle connection of the first torus isometrically to the right saddle connection of the second torus, then glue the left saddle connection of the second torus isometrically to the right saddle connection of the third torus and so forth. Finally glue the left saddle connection of the last torus isometrically to the right saddle connection of the first torus. All the top ends of the slits should be identified to a point as well as all the bottom ends (we call this operation gluing slit tori around a slit). We get a flat surface with no boundary with two singular points p_1 and p_2 of order a_2 each, and assume that p_1 is the top one. Second, on one of the saddle connections that exist on the flat surface, mark a (regular) point p_3 (above p_2 and below p_1). Cut the surface along the vertical line between p_1 and p_3. Around the slit, we glue a_3 minimal slit tori in the same fashion. Now our flat surface has three singular points p_1, p_3 and p_2 (top to bottom) of orders a_2+a_3, a_3, a_2 respectively. We continue in this way until we get a flat surface in $\mathcal{H}$ with j singular points of orders $\sum_{i=2}^{j} a_i = g-1, a_j, a_{j-1}, \ldots, a_2$. This surface is of genus g and it is easy to see from the construction that it has g minimal components.

If $a_k = g-1$ or $a_l = g-1$ there is always a saddle connection between the singular points p_k and p_l, as there are saddle connections between the singular point of order $g-1$ and any other singular point. Otherwise, just permute the set $\{a_2, a_3, \ldots, a_j\}$ so a_k and a_l are adjacent. Note that the stratum $\mathcal{H}(g-1, g-1)$ is included in this case.

- $\mathcal{H}$ is of the form $\mathcal{H}(a_1, a_2, a_3, \ldots, a_j)$ where $a_i < g-1$ for all i.

  Look at the stratum

$$\mathcal{H}'(a_1, \ldots, a_{k-1}, a_k - 1, a_{k+1}, \ldots, a_{l-1}, a_l - 1, a_{l+1}, \ldots, a_j)$$

over a surface of genus $g-1$ (it is possible that $a_k - 1 = 0$ or $a_l - 1 = 0$). Now, $a_i \leq g-2,\ i = 1, 2, \ldots, j$ and by the induction hypothesis there exists a flat surface (S', Φ') in $\mathcal{H}'$ with $g-1$ minimal components and a saddle connection between the singular points p_k and p_l of orders $a_k - 1$, $a_l - 1$, respectively. Cut along this saddle connection and glue to the slit a minimal slit torus. This results in a flat surface (S, Φ) in $\mathcal{H}$ with g minimal components and the required saddle connection.

If $a_k - 1 = 0$ and $a_l - 1 = 0$, just mark a vertical segment on (S', Φ'). Cut along this segment and glue to the slit a minimal slit torus.

If $a_k - 1 = 0$ and $a_l - 1 > 0$ (or the symmetric case), look at the singular point p_l on (S', Φ'). According to Lemma 3.2 there exist a minimal component L with p_l being a top or bottom end of a slit in the boundary of L. So we can always mark a vertical segment in the interior of L starting at p_l. Cut along this segment and glue to the slit a minimal slit torus. ∎

The second case of Theorem 1 is a corollary of Proposition 3.4 below (the case where $\mathbf{M} = g-1$).

Before we start with the bounds in Theorem 2, we present two more building blocks coming from zippered rectangles that we will use.

P3: This is a flat surface of genus 2 in $\mathcal{H}(1,1)$ realized via a zippered rectangles construction. The parameters are:

The permutation Π

$$\Pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 2 & 1 \end{pmatrix}$$

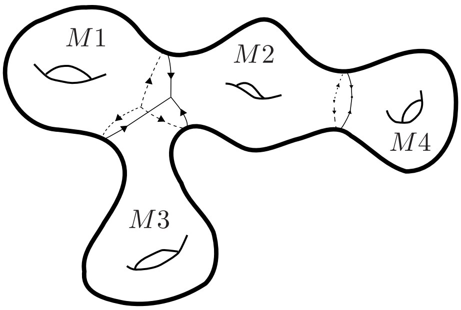


Figure 3. A genus 4 flat surface with 4 minimal components, all minimal tori with slits. This flat surface lies in $\mathcal{H}(2,2,1,1)$.

the 3 n-tuples are

$$X = \left(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right) \quad h = \left(1, \frac{3}{2}, 1, \frac{3}{2}, 1\right) \quad a = \left(\frac{1}{2}, \frac{1}{2}, 0, 1, -\frac{1}{2}\right)$$

By Veech's construction, this data give rise to a flat surface in $\mathcal{H}(1,1)$, which we denote by $P3$. $P3$ has three periodic components (can easily be seen by looking at Π and X). The lengths of the periodic orbits are 1, 2 and 3.

P2: This is a flat surface of genus 2 in $\mathcal{H}(2)$ realized via a zippered rectangles construction. The parameters are:

The permutation Π

$$\Pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

the 3 n-tuples are

$$X = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right) \quad h = (1, 1, 1, 1) \quad a = (1, 0, 1, 0)$$

This data give rise to a flat surface in $\mathcal{H}(2)$, which we denote by $P2$. $P2$ has two periodic components (can easily be seen by looking at Π and X).

The next proposition proves that the bound stated in Theorem 2 is tight in every stratum in the case of totally periodic surfaces, that is, surfaces with periodic components only (and no minimal components).

Proposition 3.3: *Let $\mathcal{H} = \mathcal{H}(a_1, a_2, \ldots, a_j)$ be a stratum of surfaces of genus $g \geq 2$. Then there exists a flat surface in $\mathcal{H}$ with $g - 1 + j$ periodic components (and no minimal components). Moreover ,if $\mathcal{H}$ is the principal stratum then there exists a periodic component with orbit length 2.*

Proof. First we address the case where $\mathcal{H}$ is the principal stratum, that is $\mathcal{H}(\underbrace{1, 1, \ldots, 1}_{j})$. The proof will be by induction on the number of singular points j (which is always even). The building block $P3$ proves the case where $j = 2$ as $g - 1 + j = 3$. $P3$ has a periodic component of length 2.

In the general case, Look at the stratum $\mathcal{H}'(\underbrace{1, 1, \ldots, 1}_{j-2})$. This stratum lies over a surface of genus $g - 1$. By the induction hypothesis there exists a flat surface (S', Φ') in $\mathcal{H}'$ with $(g - 1) - 1 + (j - 2) = g - 1 + j - 3$ periodic components. Cut (S', Φ') along a closed vertical geodesic in the interior of the periodic component with orbit length 2. Cut $P3$ along a closed vertical geodesic in the interior of the periodic component with orbit length 2. The two surfaces stay connected after the cuts as the curves are not homologous to zero. Now identify isometrically one boundary curve of (S', Φ') to one boundary curve of $P3$ and do the same with the remaining boundary curves. This results in a flat surface in $\mathcal{H}$, with 3 additional periodic components. So this surface has $g - 1 + j$ periodic components as desired.

Now assume that $\mathcal{H}(a_1, a_2, \ldots, a_j)$ is not the principal stratum. If we assume $a_1 \leq a_2 \leq \cdots \leq a_j$ then $2 \leq a_j$. Again, we will use induction but now the induction is on the genus g. If $g = 2$, the only stratum to consider is $\mathcal{H}(2)$. Our building block $P2$ realizes the bound in this case as $g - 1 + j = 2$.

In the general case, let $\mathcal{H}(a_1, a_2, \ldots, a_j)$ be a stratum over a surface of genus $3 \leq g$. Consider $\mathcal{H}'(a_1, a_2, \ldots, a_j - 2)$ over a surface of genus $g' = g - 1$. There are two cases to consider

- $a_j - 2 = 0$.

  Then $\mathcal{H}'$ can be written $\mathcal{H}'(a_1, a_2, \ldots, a_{j-1})$. According to the induction hypothesis, there is a totally periodic flat surface in $\mathcal{H}'$ with $g' - 1 + j' = g - 1 + j - 2$ periodic components. Let p be a regular point in the interior of a periodic component, and let us cut along the interior of the closed vertical geodesic starting (end ending) at p. We now take a periodic figure-8 torus, and identify its boundary to the boundary of our surface to get a closed surface. This process added two periodic components; one is the torus that was added and the second comes from splitting the periodic component in which p was chosen into two periodic components. The resulting surface is in $\mathcal{H}$ and has $g-1+j$ periodic components.

- $a_j - 2 > 0$.

  By the induction hypothesis, there is a totally periodic flat surface in $\mathcal{H}'$ with $g'-1+j' = g-1+j-1$. Let p by the singular point corresponding to the $a_j - 2$ order zero. Now we cut along the interior of the vertical circle starting (and ending) at p and again close the resulting boundary with the boundary of a periodic figure-8 torus, to get a flat surface in $\mathcal{H}$ with $g-1+j$ periodic components. ∎

Now we prove the general case of Theorem 2.

Proposition 3.4: *Let $\mathcal{H} = \mathcal{H}(a_1, a_2, \ldots, a_j)$ be a stratum of surfaces of genus $g \geq 2$. Denote $B_1 = \{i : a_i \text{ is even}\}$ and $B_2 = \{i : a_i \text{ is odd}\}$. Fix $0 \leq \mathbf{M} \leq g-1$ and denote $m = \max\{0, \mathbf{M} - [g-1-|B_2|/2]\}$, then:*

1. *If $\mathbf{M} \leq g-1-|B_2|/2$ then there exists a flat surface in $\mathcal{H}$ with $\mathbf{M}$ minimal components and $\mathbf{P}$ periodic components such that $\mathbf{M}+\mathbf{P} = g-1+j$.*
2. *If $g-1-|B_2|/2 < \mathbf{M} \leq g-1$ then then there exists a flat surface in $\mathcal{H}$ with $\mathbf{M}$ minimal components and $\mathbf{P}$ periodic components such that $2\mathbf{M}+\mathbf{P} = 2g-2+j-|B_2|/2$. Equivalently, $\mathbf{M}+\mathbf{P} = g-1+j-m$.*

In both cases all singular points lies on the boundary of a periodic component.

Remark 3.5: When one of the bounds is realized, $\mathbf{P}$ is always greater than zero.

Proof. The proof will be by induction on the genus g. We assume that $a_1 \leq a_2 \leq \cdots \leq a_j$. In the case of $g = 2$ there are two strata in genus 2.

- $\mathcal{H}(2)$, $m = 0$. If $\mathbf{M} = 0$ then by Proposition 3.3 the bound is met. If $\mathbf{M} = 1$, take a minimal figure-8 torus and close its boundary with a periodic cylinder. Then $\mathbf{M}+\mathbf{P} = g-1+j = 2$.
- $\mathcal{H}(1,1)$. If $\mathbf{M} = 0$ then by Proposition 3.3 the bound is met. If $\mathbf{M} = 1$ ($m = 1$) then $2g-2+j-\frac{|B_2|}{2} = 3$. Take a minimal slit torus and identify its boundary with the boundary of a periodic slit torus - $2\mathbf{M}+\mathbf{P} = 3$.

In all the constructions, all singular points are on the boundary of a periodic component.

For the general case, we can assume that $\mathbf{M} \neq 0$ (as this case is dealt by Proposition 3.3). First we consider the case of the principal stratum where $\mathcal{H}(\underbrace{1,1,\ldots,1}_{j})$. A simple computation yields $2g-2+j-\frac{|B_2|}{2} = g-1+j$ (and $m = \mathbf{M}$). Look at the stratum $\mathcal{H}'(\underbrace{1,1,\ldots,1}_{j-2})$. This stratum lies over a surface of genus $g' = g-1$. If $m = 1$, by Proposition 3.3 there exists a totally periodic flat surface in $\mathcal{H}'$ with $\mathbf{P}' = g'-1+(j-2)$. Slit the interior of a periodic component and identify the boundary with the boundary of a minimal slit torus. We added one periodic and one minimal component so now $2\mathbf{M}+\mathbf{P} = 3+\mathbf{P}' = g-1+j$. Otherwise, $\mathbf{M} > 1$ and by the induction hypothesis, there exists a flat surface in $\mathcal{H}'$ with $\mathbf{M}' = \mathbf{M}-1$ minimal components and $\mathbf{P}'$ periodic components such that $2\mathbf{M}'+\mathbf{P}' = g'-1+(j-2)$. Perform the same construction to get a flat surface admitting $2\mathbf{M}+\mathbf{P} = 2\mathbf{M}'+\mathbf{P}'+3 = g-1+j$.

In the remaining case $2 \leq a_j$. Consider the stratum $\mathcal{H}'(a_1, a_2, \ldots, a_j - 2)$ over a surface of genus $g' = g-1$ and $|B_2| = |B_2'|$. As in the proof of Proposition 3.3 there are two cases:

- $a_j - 2 = 0$.

  Then $\mathcal{H}'$ can be written $\mathcal{H}'(a_1, a_2, \ldots, a_{j-1})$. According to the induction hypothesis (or Proposition 3.3 in the case where $\mathbf{M} = 1$), there exists a flat surface in $\mathcal{H}'$ with $\mathbf{M}' = \mathbf{M}-1$ and $\mathbf{P}'$ such that the bound is realized. Note that $\mathbf{M} \leq g-1-|B_2|/2$ if and only if $\mathbf{M}' \leq g'-1-|B_2'|/2$. Let p be a regular point in the interior of a periodic component, and let us cut along the interior of the closed vertical geodesic starting (end ending) at p. We now take a minimal figure-8 torus, and identify its boundary to the boundary of our surface to get a closed surface. This process added one minimal component and one periodic component.

  If $\mathbf{M}' \leq g'-1-|B_2'|/2$, then

  $$\mathbf{M}+\mathbf{P} = \mathbf{M}'+\mathbf{P}'+2 = g'-1+(j-1)+2 = g-1+j.$$

  Otherwise $\mathbf{M}' > g'-1-|B_2'|/2$ and

  $$\begin{aligned} 2\mathbf{M}+\mathbf{P} &= 2\mathbf{M}'+\mathbf{P}'+3 = 2g'-2+(j-1)-\frac{|B_2'|}{2}+3 \\ &= 2g-2+j-\frac{|B_2|}{2}. \end{aligned}$$

- $a_j - 2 > 0$.

According to the induction hypothesis (or Proposition 3.3 in the case where $\mathbf{M} = 1$), there exists a flat surface in $\mathcal{H}'$ with $\mathbf{M}' = \mathbf{M} - 1$ and $\mathbf{P}'$ such that the bound is realized. Again it is true that $\mathbf{M} \leq g - 1 - |B_2|/2$ if and only if $\mathbf{M}' \leq g' - 1 - |B_2'|/2$. Let p by the singular point corresponding to the $a_j - 2$ order zero. We cut along the interior of the vertical circle starting (and ending) at p, which is a boundary of a periodic component. Again, we close the resulting boundary with the boundary of a minimal figure-8 torus, to get a flat surface in $\mathcal{H}$. If $\mathbf{M}' \leq g' - 1 - |B_2'|/2$ then

$$\mathbf{M} + \mathbf{P} = \mathbf{M}' + \mathbf{P}' + 1 = g' - 1 + j + 1 = g - 1 + j.$$

Otherwise $\mathbf{M}' > g' - 1 - \frac{|B_2'|}{2}$ and

$$\begin{aligned} 2\mathbf{M} + \mathbf{P} = 2\mathbf{M}' + \mathbf{P}' + 2 &= 2g' - 2 + j - \frac{|B_2'|}{2} + 2 \\ &= 2g - 2 + j - \frac{|B_2|}{2}. \end{aligned}$$

This completes the proof. ∎

References

[Bo] M. Boshernitzan, *Rank two interval exchange transformations*, Erg. Th. Dyn. Sys. **8** (1988), 379–394.

[EMZ] A. Eskin, H. Masur, A. Zorich *Moduli spaces of abelian differentials: the principal boundary, counting problems and the Siegel-Veech constants*, Publications Mathèmatiques. Institut de Hautes Ètudes Scientifiques **97** (2003), 61–179.

[Ke] M. Keane, *Interval Exchange Transformations*, Mathematische Zeitschrift **141** (1975), 25–31.

[MS] H. Masur, J. Smillie, *Quadratic differentials with prescribed singularities and pseudo-Anosov diffeomorphisms*, Commentarii Mathematici Helvetici **68** (1993), 289-307.

[MT] H. Masur and S. Tabachnikov, *Rational billiards and flat structures*, in *Handbook on Dynamical Systems*, Volume 1A, Elsevier Science, North Holland, 2002, pp. 1015–1089.

[Ve] W. A. Veech, *Gauss measures for transformations on the space of interval exchange maps*, Annals of Mathematics **115** (1982), 201–242.